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Solitons in supersymmetric sigma models with torsion

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ABSTRACT

We derive a bound on the energy of the general (p,q) -supersymmetric two-dimensional massive sigma model with torsion, in terms of the topological and Noether charges that appear as central charges in its supersymmetry algebra. The bound is saturated by soliton solutions of first-order Bogomol'nyi-type equations. This generalizes results obtained previously for $p = q$ models without torsion. We give examples of massive $(1,1)$ models with torsion that have a group manifold as a target space. We show that they generically have multiple vacua and find an explicit soliton solution of an $SU(2)$ model. We also construct a new class of zero torsion massive $(4,4)$ models with multiple vacua and soliton solutions. In addition, we compute the metrics on the one-soliton moduli spaces for those cases for which soliton solutions are known explicitly, and discuss their interpretation.

1. Introduction

For certain supersymmetric field theories, the algebra of supersymmetry charges can be modified in the presence of a soliton by the appearance of the soliton's topological charge as a central charge [1]. In some cases the soliton may carry a Noether charge too, and this may also appear as a central charge in the supersymmetry algebra. Such solitons are generically referred to as ‘Q-solitons’. The appearance of central charges in the supersymmetry algebra implies a bound on the energy of arbitrary field configurations of the type

$$E \geq m \sqrt{\sum Q^2} , \quad (1.1)$$

for suitably normalized charges, Q , topological or Noether, where m is a mass parameter. Bounds of this type were originally derived by Bogomol’nyi [2] for various bosonic field theories but without mention of supersymmetry. Bogomol’nyi’s arguments have a natural interpretation in terms of supersymmetry, however, and it is only in this context that the bound can be expected to survive quantum corrections. The bound is saturated by solutions to first-order equations which imply, but are much simpler than, the second-order field equations. The solutions of these first-order, Bogomol’nyi, equations are precisely the soliton, or Q-soliton, solutions. An example of a Q-soliton in a four-dimensional supersymmetric field theory that can be found in this way is the BPS dyon of $N = 2$ super Yang-Mills theory.

As is well-known, two-dimensional non-linear sigma-models are analogous in many respects to four-dimensional gauge theories. The two-dimensional analogues of the magnetic monopole and dyon are the kink and Q -kink [3] solutions of sigma models with a scalar potential, which we shall call ‘massive’ sigma-models since the presence of the scalar potential introduces a mass parameter. Omitting fermions, the action takes the form

$$I = \int dx dt \left[(g + b)_{ij} \partial_{\mp} \phi^i \partial_{\pm} \phi^j - V(\phi) \right] , \quad (1.2)$$

where ϕ is a map from the two-dimensional Minkowski space-time, with light-cone

co-ordinates (x^\pm, x^-) , into the target manifold \mathcal{M} with metric g . The two-form b is a locally-defined potential for a globally-defined ‘torsion’ three-form H with components $H_{ijk} = \frac{3}{2}\partial_{[i}b_{jk]}$, and V is the scalar potential.

The charges carried by the kink or Q -kink solutions of the (p,q)-supersymmetric sigma-model field equations appear as central charges in the (p,q)-supersymmetry algebra; note that this is true not only of the topological charges but also of the Noether charges. Lorentz-invariant central charges are possible only if neither p nor q vanishes, so the sigma-models of relevance here have at least (1,1) supersymmetry. All such models have a scalar potential of the form [4]

$$V = \frac{1}{4}m^2 g^{ij}(u - X)_i(u - X)_j , \quad (1.3)$$

where m is the mass parameter, X is a (possibly vanishing) Killing vector field on \mathcal{M} and u is a one-form on \mathcal{M} whose exterior derivative du is determined by X and H via the formula

$$X^k H_{kij} = \partial_{[i}u_{j]} . \quad (1.4)$$

In particular, $du = 0$ if either H or X vanishes in which case $u = da$ for some locally-defined scalar a on \mathcal{M} , which can be identified as the superpotential of the (1,1)-superspace formulation. If X is non-vanishing then H must be invariant under the symmetry generated by X , i.e. $\mathcal{L}_X H = 0$. The conditions $\mathcal{L}_X H = 0$ and $dH = 0$ imply only that $X \cdot u = \text{const.}$ but consideration of the closure of the Poisson bracket algebra of charges for Minkowski space-time requires that

$$X \cdot u = 0 . \quad (1.5)$$

These requirements imply that u , and hence the scalar potential V , is invariant under the symmetry generated by X .

A limitation of all models for which the soliton structure has been investigated to date is that they are non-chiral. That is, the number of left and right-handed

supersymmetries is equal, i.e. $p = q$, and there is no parity-violating Wess-Zumino, or ‘torsion’, term in the action. Recently, we have determined the conditions imposed by *on-shell* (p,q) supersymmetry[★] on the form of the potential V in the general massive sigma model with torsion [5,6], generalizing the off-shell $(p,0)$ and $(1,1)$ results of [4] and the earlier results for (p,p) models without torsion [7]. One purpose of this paper is to present some examples of massive sigma models, with and without torsion, for which these conditions are satisfied and which are therefore (p,q) -supersymmetric. In particular, we present massive extensions of the $(1,1)$ -supersymmetric WZW models and a class of $(4,4)$ models with hyper-Kähler $(4k)$ -metrics admitting k commuting tri-holomorphic Killing vector fields. The latter examples generalize the $k=1$ models based on the Gibbons-Hawking 4-metrics [8].

Our main interest in these models in this paper is the possibility of soliton solutions interpolating between isolated zeros of the potential V , an example of which is the Q-kink solution found in [3] for a $(4,4)$ supersymmetric sigma model. Using the results of [6] for the Poisson bracket algebra of supersymmetry charges, reviewed in section 2, we derive in section 3 a Bogomol’nyi bound on the energy of a general field configuration, and we show that the bound is saturated by solutions of first order equations of Bogomol’nyi type. The finite energy solutions of these equations are the soliton solutions of these models. In subsequent sections we investigate in detail various special cases and find a new explicit soliton solution in a massive $(1,1)$ models with torsion and the $SU(2)$ group as its target space. We also compute the metric on the one-soliton moduli space in both this example and the Q-kink example of [3], and hence the one-soliton low-energy effective Lagrangian. In the Q-kink example this Lagrangian has a Kaluza-Klein interpretation.

★ These conditions were found for both off-shell and on-shell supersymmetry. The conditions for off-shell supersymmetry are more restrictive than those of on-shell supersymmetry.

2. Massive supersymmetric sigma-models

We begin with a summary of those results of refs. [4,5,6] that will be relevant to the derivation of the Bogomol'nyi bound in the following section. The component action of the general (1,1)-supersymmetric model is a functional of the sigma model fields ϕ^i and the chiral fermions λ_+ and ψ_- , and takes the form

$$I = \int d^2x \left\{ \partial_{\mp} \phi^i \partial_{\pm} \phi^j (g_{ij} + b_{ij}) + i g_{ij} \lambda_+^i \nabla_{\pm}^{(+)} \lambda_+^j - i g_{ij} \psi_-^i \nabla_{\mp}^{(-)} \psi_-^j - \frac{1}{2} \psi_-^k \psi_-^l \lambda_+^i \lambda_+^j R_{ijkl}^{(-)} + m \nabla_i^{(-)} (u - X)_j \lambda_+^i \psi_-^j - V(\phi) \right\}, \quad (2.1)$$

where V is given in eqn. (1.3), and $\nabla^{(\pm)}$ are the covariant derivatives with connections

$$\Gamma_{ij}^{(\pm)k} = \left\{ \begin{smallmatrix} k \\ ij \end{smallmatrix} \right\} \pm H_{ij}{}^k; \quad (2.2)$$

i.e. H_{ijk} is the torsion of the connection of $\nabla^{(+)}$. We refer to [5,6] for details of the conventions.

All (p, q) -supersymmetric sigma models with $p, q \geq 1$ are special cases of the (1,1)-supersymmetric model. The additional supersymmetries simply impose further restrictions on the sigma model couplings and the geometry of \mathcal{M} . In the massless case these restrictions are long-established [9,10]. For example, an additional $p - 1$ left-handed supersymmetries requires the existence of $p - 1$ complex structures I_r on \mathcal{M} that are covariantly constant with respect to the connection $\Gamma^{(+)}$, and that the metric g of \mathcal{M} be hermitian with respect to them. In the case that $p = 4$, closure of the algebra of supersymmetry transformations requires in addition that the complex structures I_r ($r = 1, 2, 3$) obey the algebra of imaginary unit quaternions. Similarly, an additional $q - 1$ right-handed supersymmetries requires the existence of $(q - 1)$ -complex structures J_s on \mathcal{M} , but in this case the complex structures J_s are covariantly constant with respect to the connection $\Gamma^{(-)}$. The metric g must be Hermitian with respect to all complex structures.

Apart from the complex structures I_r and J_s , the massive (p,q)-supersymmetric sigma models also depend on a set of Killing-vector/one-form pairs $\{Z_{I'I}, u^{I'I}\}$. The Killing vector fields $Z_{I'I}$ leave the torsion H and the complex structures I_r and J_s invariant and generate new symmetries of the action, and they also have the same relationship with the one-forms $u^{I'I}$ as X has to u , i.e

$$Z_{I'I} \cdot H = du^{I'I}. \quad (2.3)$$

For consistency with [6] we adopt the notation

$$\begin{aligned} Z_{00} &= X & Z_{0r} &= Z_r & Z_{s0} &= Y_s & Z_{sr} &= Z_{sr} \\ u^{00} &= u & u^{0r} &= v_r & u^{s0} &= w_s & u^{sr} &= v_{sr} . \end{aligned} \quad (2.4)$$

The most important restrictions imposed by (p,q) supersymmetry can now be summarized by the following set of relations:

$$\begin{aligned} (Z_r + v_r)_i + I_r^k{}_i (X + u)_k &= 0 \\ (Y_s - w_s)_i + J_s^k{}_i (X - u)_k &= 0 \\ (Z_{sr} + v_{sr})_i + I_r^k{}_i (Y_s + w_s)_k &= 0 \\ (Z_{sr} - v_{sr})_i + J_s^k{}_i (Z_r - v_r)_k &= 0 \end{aligned} \quad (2.5)$$

and

$$Z_{I'I} \cdot u^{J'J} + Z_{J'J} \cdot u^{I'I} = 0 , \quad (2.6)$$

which includes (1.5) as a special case. For a discussion of the details see [6].

The supersymmetry charges of these models consist of the (1,0) and (when applicable) extended (p,0) charges, S_+ and S_+^r , and the (0,1) and extended (0,q) charges, S_- and S_-^s . The other, bosonic, charges appearing in the supersymmetry algebra are the energy, E , the momentum, P , and the central charges $Q_{I'I}$. These

charges have the following expressions[★] in terms of the sigma-model fields:

$$\begin{aligned}
E &= \frac{1}{2} \int dx [g_{ij} \partial_t \phi^i \partial_t \phi^j + g_{ij} \partial_x \phi^i \partial_x \phi^j + V(\phi) + \text{fermions}] \\
P &= \int dx [g_{ij} \partial_t \phi^i \partial_x \phi^j + \text{fermions}] \\
Q_{I'I} &= \int dx [Z^{I'I}{}_i \partial_t \phi^i + u^{I'I}{}_i \partial_x \phi^i + \text{fermions}] \\
S_+ &= \int dx [g_{ij} \partial_+ \phi^i \lambda_+^j - \frac{1}{3} H_{ijk} \lambda_+^i \lambda_+^j \lambda_+^k - \frac{i}{2} m(u - X)_i \psi_-^i] \\
S_- &= \int dx [i g_{ij} \partial_- \phi^i \psi_-^j + \frac{1}{3} \psi_-^i \psi_-^j \psi_-^k H_{ijk} + \frac{m}{2} (X + u)_i \lambda_+^i] \\
S_+^r &= \int dx [I_r{}_{ij} (\partial_+ \phi^i \lambda_+^j - i H^i{}_{kl} \lambda_+^k \lambda_+^l) - i \frac{m}{2} (v_r - Z_r)_i \psi_-^i] \\
S_-^s &= \int dx [i J_s{}_{ij} \partial_- \phi^i \psi_-^j + \frac{1}{3} H_{mnl} J_s{}^m{}_i J_s{}^n{}_j J_s{}^l{}_k \psi_-^i \psi_-^j \psi_-^k + \frac{m}{2} (Y_s + w_s)_i \lambda_+^i] .
\end{aligned} \tag{2.7}$$

The fermion contributions to the bosonic charges will not be needed for what follows so we have omitted them.

Rewriting the above charges in terms of the sigma model fields and their conjugate momenta, and using the canonical Poisson bracket relations, one finds the following Poisson Bracket algebra of the charges:

$$\begin{aligned}
\{S_+, S_+\} &= 2(E + P), \quad \{S_+^r, S_+^s\} = 2\delta_{rs}(E + P), \quad \{S_+, S_+^r\} = 0, \\
\{S_-, S_-\} &= 2(E - P), \quad \{S_-^r, S_-^s\} = 2\delta_{rs}(E - P), \quad \{S_-, S_-^s\} = 0, \\
\{S_+, S_-\} &= mQ_{oo}, \quad \{S_+, S_-^s\} = mQ_{so}, \\
\{S_-, S_+^r\} &= mQ_{or}, \quad \{S_+^r, S_-^s\} = mQ_{sr} .
\end{aligned} \tag{2.8}$$

In addition, the Poisson brackets of the $Q_{I'I}$ amongst themselves, and with all supersymmetry charges, vanish, i.e. the $Q_{I'I}$ are central charges of the supersymmetry algebra.

★ This corrects the omission in [6] of a torsion term in S_+ .

3. Bogomol'nyi Bounds

Using the algebra of Poisson brackets just reviewed, the energy E can be written as

$$E = \{\alpha_+ S_+ - \alpha_- S_- + \beta_{+r} S_+^r - \beta_{-s} S_-^s, \alpha_+ S_+ - \alpha_- S_- + \beta_{+r} S_+^r - \beta_{-s} S_-^s\} + m \left(\alpha_+ \alpha_- Q_{oo} + \alpha_+ \beta_-^s Q_{so} + \alpha_- \beta_+^r Q_{or} + \beta_+^r \beta_-^s Q_{sr} \right) \quad (3.1)$$

where $\{\alpha_-, \alpha_+, \beta_-^s, \beta_+^r\}$ are real constants restricted by

$$\alpha_+^2 + \sum_{r=1}^{p-1} (\beta_+^r)^2 = \frac{1}{2} \quad \alpha_-^2 + \sum_{s=1}^{q-1} (\beta_-^s)^2 = \frac{1}{2}. \quad (3.2)$$

Using in (3.1) the expressions given in (2.7) for the supersymmetry charges, and evaluating the resulting expression by using the canonical Poisson brackets of the fields with their conjugate momenta, we find (omitting fermion terms) that

$$E = \int dx \left[g_{ij} \left(\partial_{\mp} \phi^i - m(\alpha_+ \delta_k^i + \beta_+^r I_r^i{}_k)(\alpha_- (X + u)^k + \beta_-^s (Y_s + w_s)^k) \right) \right. \\ \left(\partial_{\mp} \phi^j - m(\alpha_+ \delta_l^j + \beta_+^r I_r^j{}_l)(\alpha_- (X + u)^l + \beta_-^s (Y_s + w_s)^l) \right) + \\ g_{ij} \left(\partial_{\pm} \phi^i + m(\alpha_- \delta_k^i + \beta_-^s J_s^i{}_k)(\alpha_+ (u - X)^k + \beta_+^r (v_r - Z_r)^k) \right) \\ \left(\partial_{\pm} \phi^j + m(\alpha_- \delta_l^j + \beta_-^s J_s^j{}_l)(\alpha_+ (u - X)^l + \beta_+^r (v_r - Z_r)^l) \right) \Big] + \\ m \left(\alpha_+ \alpha_- Q_{oo} + \alpha_+ \beta_-^s Q_{so} + \alpha_- \beta_+^r Q_{or} + \beta_+^r \beta_-^s Q_{sr} \right) \quad (3.3)$$

where $(\partial_{\mp}, \partial_{\pm}) = (\partial_t + \partial_x, \partial_t - \partial_x)$.

An immediate consequence of (3.3) is that

$$E \geq \frac{m}{2} \xi \cdot \mathcal{Q} \quad (3.4)$$

where $\xi = 2(\alpha_- \alpha_+, \alpha_+ \beta_-^s, \alpha_- \beta_+^r, \beta_-^s \beta_+^r)$ is a (pq) -vector of unit length (as result of (3.2)) and $\mathcal{Q} = (Q_{oo}, Q_{so}, Q_{or}, Q_{sr})$. By choosing ξ parallel to \mathcal{Q} we derive the

energy bound

$$E \geq \frac{1}{2}|m|\sqrt{|Q_{oo}|^2 + \sum_s |Q_{so}|^2 + \sum_r |Q_{or}|^2 + \sum_{s,r} |Q_{sr}|^2} \quad (3.5)$$

which is saturated by solutions of the first-order, ‘Bogomol’nyi’ equations

$$\partial_t \phi^i = \mathcal{X}^i, \quad \partial_x \phi^i = \mathcal{Y}^i, \quad (3.6)$$

where \mathcal{X} and \mathcal{Y} are the vector fields

$$\begin{aligned} \mathcal{X}^i &\equiv m(\alpha_+ \alpha_- X^i + \alpha_+ \beta_-^s Y_s^i + \alpha_- \beta_+^r Z_r^i + \beta_+^r \beta_-^s Z_{sr}^i) \\ \mathcal{Y}^i &\equiv m(\alpha_+ \alpha_- u^i + \alpha_+ \beta_-^s w_s^i + \alpha_- \beta_+^r v_r^i + \beta_+^r \beta_-^s v_{sr}^i) \\ 1 \leq r \leq p \quad 1 \leq s \leq q. \end{aligned} \quad (3.7)$$

It can be shown, using the eqns. (2.6), that the vector fields \mathcal{X} and \mathcal{Y} are orthogonal and commute under Lie brackets. Therefore the flows of the vector fields \mathcal{X} and \mathcal{Y} commute and are orthogonal. The most general solution of (3.6) is given by

$$\phi(x, t) = \mathcal{F}^{\mathcal{X}}_t(\mathcal{F}^{\mathcal{Y}}_x(p)) = \mathcal{F}^{\mathcal{Y}}_x(\mathcal{F}^{\mathcal{X}}_t(p)), \quad p \in \mathcal{M}, \quad (3.8)$$

where $\mathcal{F}^{\mathcal{X}}, \mathcal{F}^{\mathcal{Y}}$ are the flows of the vector fields \mathcal{X} and \mathcal{Y} respectively such that $\mathcal{F}^{\mathcal{X}}_o(p) = p$ and $\mathcal{F}^{\mathcal{Y}}_o(p) = p$. Therefore the different solutions of (3.6) are parameterised by the points p of the sigma model target manifold \mathcal{M} . Note that *both* vector fields \mathcal{X} and \mathcal{Y} vanish at the points $\{p_0\}$ for which the potential V vanishes, i.e. at the classical vacua; in this case $\phi(x, t) = p_0$. Other finite energy solutions include those which interpolate between these classical vacua, but those solutions which merely interpolate between two zeros of \mathcal{Y} also have finite energy. In some cases there are ‘cosmological’ solutions for which ϕ depends only on time.

A metric can be defined on the space of solutions, ϕ , of the Bogomol'nyi equations as

$$ds^2 = \frac{1}{2} \int dx g_{ij}(\phi^i) d\phi^i d\phi^j, \quad (3.9)$$

where $d\phi$ is a differential of ϕ with respect to the parameters of the solutions. Using that \mathcal{X} is a Killing vector field, so its flow leaves invariant the sigma model metric g , the metric (3.9) can be rewritten as

$$ds^2 = \frac{1}{2} \int dx (\mathcal{F}_x^{\mathcal{Y}*} g)_{ij}(p) dy^i(p) dy^j(p) \quad (3.10)$$

where y^i is a set of coordinates on \mathcal{M} at $\phi(0,0) = p$. The metric ds^2 is not well-defined for all solutions of the Bogomol'nyi equations because the expression in equation (3.10) can become infinite, but it is well-defined for soliton solutions, as we shall confirm.

4. (1,1) Models: the massive supersymmetric WZW model.

For (1,1) models eqn. (3.7) becomes

$$\partial_x \phi^i \pm \frac{m}{2} u^i = 0 \quad \partial_t \phi^i \pm \frac{m}{2} X^i = 0. \quad (4.1)$$

As mentioned in the previous section, these equations describe the flows of two orthogonal commuting vector fields X and u , so the two equations are simultaneously integrable. The solutions of these equations are $\phi(x,t) = \mathcal{F}_t^X \mathcal{F}_x^u(p)$ ($p \in \mathcal{M}$), and those with finite energy must interpolate between zeros of u . Those solutions, parametrised by the points $p \in \mathcal{M}$, for which the vector field X vanishes, are static.

When the torsion H vanishes everywhere on \mathcal{M} (but not, necessarily, X) the first equation in (2.3) becomes $\partial_x \phi^i \pm \frac{m}{2} g^{ij} \partial_j a = 0$ where $u = da$. The Noether

charge

$$Q_X = \int dx (X_i \partial_t \phi^i + u_i \partial_x \phi^i) \quad (4.2)$$

associated to X is now naturally expressed as the sum of a new Noether charge

$$Q'_X = \int dx X_i \partial_t \phi^i \quad (4.3)$$

and a topological charge

$$Q_a = \int dx \partial_x \phi^i \partial_i a . \quad (4.4)$$

For soliton solutions the value of the latter is $a(p_1) - a(p_2)$ where p_1 and p_2 are the two critical points of a between which the solution interpolates.

When the Killing vector field X vanishes everywhere on \mathcal{M} (but not, necessarily, H) $u = da$ for some (locally) defined function a . The finite energy solutions then interpolate between critical points of a , which may be identified as the superpotential of the (1,1)-superspace formulation of the model. The charge Q_X is topological and its value is given by $Q_X = a(p_1) - a(p_2) \equiv Q_a$, where p_1 and p_2 are the critical points of a that are interpolated by the solution. In principle, there might be more than one solution interpolating between a given two critical points, or none. When a is a Morse function we can define as ‘elementary’ solitons those that interpolate between critical points with Morse indices that differ by one.

Examples of models with both X and H non-vanishing may be found by consideration of sigma models with a group manifold as their target space. Let K be a group manifold with Lie algebra $\mathcal{L}(K)$ and let L^A and R^A , defined by

$$k^{-1} dk = L^A t_A \quad dk k^{-1} = R^A t_A, \quad k \in K , \quad (4.5)$$

be the left and right invariant frames, respectively, where $\{t_A\}$ is a basis in $\mathcal{L}(K)$, $[t_A, t_B] = f_{AB}^C t_C$, and f_{AB}^C are the structure constants. Note the Maurer-Cartan

equations satisfied by the frames:

$$dL^A = -\frac{1}{2}f_{BC}{}^A L^B L^C \quad dR^A = \frac{1}{2}f_{BC}{}^A R^B R^C . \quad (4.6)$$

We choose the sigma model metric g and torsion H to be the bi-invariant tensors

$$\begin{aligned} g_{ij} &= h_{AB} L_i^A L_j^B = h_{AB} R_i^A R_j^B \\ H_{ijk} &= -\frac{\lambda}{2} f_{ABC} L_i^A L_j^B L_k^C = -\frac{\lambda}{2} f_{ABC} R_i^A R_j^B R_k^C , \end{aligned} \quad (4.7)$$

where h_{AB} is a non-degenerate invariant quadratic form on $\mathcal{L}(K)$, $f_{ABC} = f_{AB}{}^D h_{DC}$ and λ is a real number. We choose the Killing vector field X such that

$$X^i = \kappa^A (L_A^i - R_A^i) , \quad (4.8)$$

where κ^A are the components of $\kappa = \kappa^A t_A$ in $\mathcal{L}(K)$. This vector field is generated by the adjoint action,

$$k \rightarrow e^{y\kappa} k e^{-y\kappa} , \quad (4.9)$$

of the one-parameter subgroup of K parametrized by y . The one-form u defined by (1.4) is then given by

$$u = \lambda \kappa_A (L^A + R^A) + da \quad (4.10)$$

where a is a function on K (possibly defined only locally) and $\kappa_A = h_{AB} \kappa^B$. The orthogonality relation $X \cdot u = 0$ (eqn (2.6)) now implies that

$$X^i \partial_i a = 0 . \quad (4.11)$$

Since it is clearly possible to find invariant functions a on K , for example the trace of any group element, we have now established the existence of (1,1) models with

torsion having a scalar potential

$$V = \frac{m^2}{4} \left(\kappa_A [(\lambda - 1)L^A + (\lambda + 1)R^A] + da \right)^2 . \quad (4.12)$$

We note for future reference that the fermion mass matrix $M_{ij} = m \nabla_i^{(-)}(u - X)_j$ is given by

$$M_{ij} = m \nabla_i^{(-)} \left(\kappa_A [(\lambda - 1)L_j^A + (\lambda + 1)R_j^A] + \partial_j a \right) . \quad (4.13)$$

For $|\lambda| = 1$ the connection $\Gamma^{(\pm)}$ is the parallellizing connection with respect to the left (+) or right (-) invariant frames, so this model is then a massive extension of the massless WZW model for the group K .

Our main interest here is in models for which the scalar potential V has multiple supersymmetric vacua; for the massive WZW models just constructed these are the points k of K for which $V = 0$. When $\lambda = 1$ they are given by

$$2\kappa_A R^A + da = 0 . \quad (4.14)$$

When a is chosen to be

$$a = \frac{1}{n} \text{tr } k^n , \quad (4.15)$$

where n is a positive integer, (4.14) becomes

$$2\kappa_A + \text{tr } k^n t_A = 0 . \quad (4.16)$$

Since κ_A is part of the definition of the model, this equation is to be solved for the group element k . We shall now investigate this equation for the group $K = SU(2)$.

Let

$$k^n = \begin{pmatrix} A & B \\ -\bar{B} & \bar{A} \end{pmatrix}; \quad A\bar{A} + B\bar{B} = 1. \quad (4.17)$$

Then, for a standard choice of basis for $\mathcal{L}(SU(2))$, (4.16) becomes

$$\begin{aligned} 2\kappa_1 + i(A - \bar{A}) &= 0 \\ 2\kappa_2 + i(B - \bar{B}) &= 0 \\ 2\kappa_3 + (B + \bar{B}) &= 0 \end{aligned} \quad (4.18)$$

for which the solution is

$$\begin{aligned} k^n &= \mp \sqrt{1 - |\kappa|^2} \mathbf{1} + \kappa^A t_A \\ &= i \begin{pmatrix} (\pm i \sqrt{1 - |\kappa|^2} + \kappa_1) & (\kappa_2 + i\kappa_3) \\ (\kappa_2 - i\kappa_3) & (\pm i \sqrt{1 - |\kappa|^2} - \kappa_1) \end{pmatrix}. \end{aligned} \quad (4.19)$$

where

$$|\kappa|^2 = \kappa_1^2 + \kappa_2^2 + \kappa_3^2. \quad (4.20)$$

There are three cases to consider: Case (i) $|\kappa| > 1$: the matrix k^n is not unitary and hence there are no supersymmetric vacua. Thus, supersymmetry is spontaneously broken. This shows, incidentally, that the Witten index vanishes. This was to be expected because the Witten index of a (1,1)-supersymmetric sigma model is the Euler number of the target manifold which vanishes for a group manifold. Case (ii) $|\kappa| = 1$: the matrix k^n is in $SU(2)$ and (by diagonalization) one can prove that there are exactly n supersymmetric vacua, counting multiplicities. In particular, there is precisely one supersymmetric vacuum state for $n = 1$ and so one state of zero energy. This result is consistent with the vanishing of the Witten index because all these vacua are degenerate in the sense that the fermion mass matrix (in the vacuum) is not invertible, so some fermions remain massless. To see this

note first that when $\lambda = 1$ (4.13) simplifies to

$$M_{ij} = m \nabla_i^{(-)} \partial_j a . \quad (4.21)$$

For the choice of a leading to the vacua k given by (4.19) this becomes

$$M_{ij} = m R_i^A R_j^B \operatorname{tr} (t_A t_B k^n) \quad (4.22)$$

and given the form of k^n in (4.19) it is easily verified that $\det M$ vanishes if and only if $|\kappa| = 1$. Case (iii) $|\kappa| < 1$: the matrix k^n is again in $SU(2)$ and (again by diagonalization) one can show that there are exactly $2n$ points k in K , counting multiplicity, for which $V = 0$, and hence $2n$ (non-degenerate) supersymmetric vacua. In particular there are precisely two supersymmetric vacua for $n = 1$. We remark that this case is similar to what happens for $\lambda = 0$, i.e. vanishing torsion, in that there too one finds $2n$ supersymmetric vacua for $a = \frac{1}{n} \operatorname{tr} k^n$.

We now turn to the Bogomol'nyi equations (4.1) for the massive WZW models considered above. These are

$$\begin{aligned} \partial_t k(x, t) \pm \frac{m}{2} (\kappa k - k \kappa) &= 0 \\ \partial_x k(x, t) \pm \frac{m}{2} \left[\kappa k + k \kappa + k \sum_A t_A \operatorname{tr} (t_A k^n) \right] &= 0 . \end{aligned} \quad (4.23)$$

The first of these equations can be solved by setting

$$k(x, t) = e^{\mp \frac{m}{2} \kappa t} k(x) e^{\pm \frac{m}{2} \kappa t} . \quad (4.24)$$

The time dependence now cancels from the second equation which reduces to the ordinary differential equation

$$\frac{dk(x)}{dx} \pm \frac{m}{2} \left[\kappa k(x) + k(x) \kappa + k(x) \sum_A t_A \operatorname{tr} (t_A k^n(x)) \right] = 0 \quad (4.25)$$

for $k(x)$, the finite energy solutions of which are the solitons.

We shall give an example of a soliton solution for $K = SU(2)$ with $a = \text{tr } k$, i.e. $n=1$, and

$$\kappa = (0, \kappa, 0); \quad |\kappa| < 1 . \quad (4.26)$$

From (4.19) one sees that the supersymmetric vacua in this case are

$$k = i \begin{pmatrix} \pm i \sqrt{1 - \kappa^2} & \kappa \\ \kappa & \pm i \sqrt{1 - \kappa^2} \end{pmatrix} , \quad (4.27)$$

To find soliton solutions we parametrise $k(x)$ as in (4.17), i.e.

$$k(x) = \begin{pmatrix} A(x) & B(x) \\ -\bar{B}(x) & \bar{A}(x) \end{pmatrix}; \quad A\bar{A} + B\bar{B} = 1 . \quad (4.28)$$

Equation (4.25) can now be rewritten as

$$\begin{aligned} -\frac{d}{dx}(A - \bar{A}) \pm \frac{m}{2}(A + \bar{A})(A - \bar{A}) &= 0 \\ -\frac{d}{dx}(B + \bar{B}) \pm \frac{m}{2}(A + \bar{A})(B + \bar{B}) &= 0 \\ \frac{d}{dx}(A + \bar{A}) \pm \frac{m}{2}[2i\kappa(B - \bar{B}) + 4 - (A + \bar{A})^2] &= 0 \\ \frac{d}{dx}(B - \bar{B}) \pm \frac{m}{2}[2i\kappa(A + \bar{A}) - (A + \bar{A})(B - \bar{B})] &= 0 . \end{aligned} \quad (4.29)$$

The solutions of these equations that interpolate between the two vacua are

$$\begin{aligned} A &= \sqrt{1 - \kappa^2} \left\{ -\tanh \left(\pm m \sqrt{1 - \kappa^2} (x - x_0) \right) + i \cos(\theta) [\cosh (m \sqrt{1 - \kappa^2} (x - x_0))]^{-1} \right\} \\ B &= i\kappa + \sqrt{1 - \kappa^2} \sin(\theta) [\cosh (m \sqrt{1 - \kappa^2} (x - x_0))]^{-1} , \end{aligned} \quad (4.30)$$

where (x_0, θ) , with ranges $-\infty < x_0 < +\infty$, $0 \leq \theta < 2\pi$, parameterize the solutions. That these solutions interpolate between the vacua (4.27) can be verified by analysis of their asymptotic behaviour as $x \rightarrow \pm\infty$. To obtain the full solution $k(x, t)$ of the Bogomol'nyi equations, we simply substitute $k(x)$ of (4.30) into (4.24)

with the vector κ given by (4.26). A computation of the charge Q_X for these solitons reveals that

$$Q_X = \mp 4(1 - \kappa^2)^{\frac{3}{2}} \quad (4.31)$$

which means that the energy is

$$E = 2m(1 - \kappa^2)^{\frac{3}{2}} \quad (4.32)$$

as can be verified explicitly. Because the two supersymmetric vacua degenerate at $|\kappa| = 1$ the soliton solution reduces to a constant at these values of κ , and the energy must vanish, as indeed it does.

The moduli space of the soliton solutions found above is topologically a cylinder because the solutions depend on the angle θ in addition to their position, x_0 , in space. The 2-metric ds^2 of (3.9) on the moduli space is

$$ds^2 = \frac{1}{m} \sqrt{1 - \kappa^2} d\theta^2 + m(1 - \kappa^2)^{3/2} dx_0^2. \quad (4.33)$$

Defining the new variable y by

$$\theta = m\sqrt{1 - \kappa^2} y \quad (4.34)$$

we observe that

$$\left(\frac{ds}{dt}\right)^2 = \frac{1}{2}E(\dot{x}_0^2 + \dot{y}^2), \quad (4.35)$$

which can be interpreted as the bosonic part of the effective low-energy worldline Lagrangian for the soliton (with mass equal to E , as expected). The full effective Lagrangian will include the coefficients of the fermionic zero modes in the soliton background as worldline fermions, and will have an $N = 1$ worldline supersymmetry.

5. (2,2) Models

For (2,2) models the Bogomol'nyi equations (3.7) become

$$\begin{aligned}\partial_t \phi^i - m(\alpha_+ \alpha_- X^i + \alpha_+ \beta_- Y^i + \alpha_- \beta_+ Z^i + \beta_+ \beta_- T^i) &= 0 \\ \partial_x \phi^i - m(\alpha_+ \alpha_- u^i + \beta_+ \beta_- n^i + \alpha_+ \beta_- w^i + \alpha_- \beta_+ v^i) &= 0 ,\end{aligned}\tag{5.1}$$

where $n = v_{11}$.

In the case that all vector fields X, T, Y, Z vanish, we set $u = da, v = db, w = dc$ and $n = de$. To find non-trivial solutions of (2.5) in this case we shall suppose that I and J commute, in which case b, c, e can be expressed in terms of a as follows:

$$\begin{aligned}\partial_i b &= -I^j{}_i \partial_j a, & \partial_i c &= -J^j{}_i \partial_j a \\ \partial_i e &= (IJ)^j{}_i \partial_j a .\end{aligned}\tag{5.2}$$

The Bogomol'nyi equations then become

$$\begin{aligned}\partial_t \phi^i &= 0 \\ \partial_x \phi^i &= m[\alpha_+ \alpha_- g^{ij} + \beta_+ \beta_- (IJ)^{ij} + \alpha_+ \beta_- J^{ij} + \alpha_- \beta_+ I^{ij}] \partial_j a .\end{aligned}\tag{5.3}$$

Since I and J commute their product $\Pi \equiv IJ$ is a product structure on the sigma model manifold \mathcal{M} . We can choose co-ordinates $\{x^i\} = \{y^a, z^p\}$ on \mathcal{M} such that $\{\Pi^i{}_j\} = \{\delta^a_b, -\delta^p_q\}$. In this coordinate system

$$\{I^i{}_j\} = \{I^a_b, I^p_q\}, \quad \{J^i{}_j\} = \{-I^a_b, I^p_q\} .\tag{5.4}$$

From the Nijenhuis conditions satisfied by the complex structures I, J one gets that $I^a_b = I^a_b(y^a)$ and $I^p_q = I^p_q(z^p)$. Further analysis of the conditions (5.2) in this

co-ordinate system reveals that (locally)

$$\begin{aligned} a &= k(y^a) + l(z^p), & e &= k(y^a) - l(z^p), \\ b &= m(y^a) + f(z^p), & b &= -m(y^a) + f(z^p) \end{aligned} \quad (5.5)$$

where k, l, m, f are functions of the indicated co-ordinates which in addition satisfy

$$\partial_a k = I^b_a \partial_b m, \quad \partial_p l = I^q_p \partial_q f. \quad (5.6)$$

The second equation of (5.3) now becomes the two equations

$$\begin{aligned} \partial_x \phi^a &= m((\alpha_+ \alpha_- + \beta_+ \beta_-) g^{ab} + (\alpha_- \beta_+ - \alpha_+ \beta_-) I^{ab}) \partial_b k \\ \partial_x \phi^p &= m((\alpha_+ \alpha_- - \beta_+ \beta_-) g^{pq} + (\alpha_- \beta_+ + \alpha_+ \beta_-) I^{pq}) \partial_q l, \end{aligned} \quad (5.7)$$

which can be further simplified by introducing complex co-ordinates on the sigma model manifold with respect to the complex structures I^a_b, I^p_q . There are four topological charges Q_a, Q_b, Q_c, Q_e one for each function a, b, c, e . The values of the topological charges for soliton configurations are

$$Q_a = a(p_1) - a(p_2), \quad Q_b = b(p_1) - b(p_2), \quad Q_c = c(p_1) - c(p_2), \quad Q_e = e(p_1) - e(p_2) \quad (5.8)$$

where p_1, p_2 are any two vacua between which the soliton solution interpolates. The topological charges can also be expressed in terms of the functions l, k, m, f .

In the case that the torsion vanishes ($H = 0$), we may choose $I = J$ and $u = da, v = db, w = dc$ and $n = de$. One then can show that (2.5) implies, locally, that

$$\begin{aligned} X_i &= I^j_i \partial_j \left(\frac{c-b}{2} \right), & Y_i &= I^j_i \partial_j \left(\frac{a+e}{2} \right), \\ T &= X, & Z &= -Y, & a-e &= 2(h + \bar{h}), & b+c &= -2i(h - \bar{h}), \end{aligned} \quad (5.9)$$

where h is a holomorphic function. The equations (5.1) now become

$$\begin{aligned} \partial_t \phi^i - m[(\alpha_+ \alpha_- + \beta_+ \beta_-) X^i + (\alpha_+ \beta_- - \alpha_- \beta_+) Y^i] &= 0 \\ \partial_x \phi^i - m[(\alpha_+ \alpha_- + \beta_+ \beta_-) g^{ij} \partial_j a + (\alpha_+ \beta_- - \alpha_- \beta_+) g^{ij} \partial_j c \\ + 2(\beta_+ \beta_- - i\alpha_- \beta_+) g^{ij} \partial_j h + 2(\beta_+ \beta_- + i\alpha_- \beta_+) g^{ij} \partial_j \bar{h}] &= 0. \end{aligned} \quad (5.10)$$

As discussed in the (1,1) case, when the torsion vanishes each Noether charge Q

can be separated into a new Noether charges Q' , of the form given in (4.3), and a topological charge. So this model has two Noether charges Q'_X and Q'_Y and a total of four topological charges Q_a, Q_b, Q_c and Q_e . The values of the latter are given in (5.8). When $h = 0$ two of the four topological charges are linearly independent and the supersymmetry algebra has an $O(2)$ automorphism group, as discussed in more detail in [6]. Thus, the $O(2)$ invariant models have two Noether and two topological charges and are hence are possible candidates for self-dual models invariant under the interchange of Noether and topological charges, as has been suggested [1] in the context of four-dimensional supersymmetric Yang-Mills theory.

A possible realization of (2,2) models for which neither the torsion vanishes nor all the Killing vectors would be the extension to (2,2) supersymmetry of the massive group manifold models that we discussed in the previous section. However, even if one could construct such a model there are good reasons for believing that the potential could not have any supersymmetric vacua, i.e. zeros, because for (2,2) models the Witten index is expected to be the sum of the number of (non-degenerate) supersymmetric vacua, as is the case for the (2,2) Landau-Ginsburg models. On this basis one would conclude that the Witten index could not vanish if there were to exist a potential V with any point for which $V = 0$, but the Witten index must vanish when the target space is a group manifold. In fact, the models discussed in the previous section cannot be extended to (2,2) supersymmetry for any choice of group manifold as the target space, despite the fact that there exist massless (2,2) supersymmetric sigma models on group manifolds for non-zero torsion.

6. (4,4) Models

To study the equations (3.7) for sigma models with (4,4) supersymmetry we first consider the case where all vector fields X, Z_r, Y_s, Z_{sr} vanish. In this case we set

$$u = da, \quad v = d, \quad w = dc, \quad v_{rr} = de_r, \quad v_{sr} = de_{sr} \quad s \neq r. \quad (6.1)$$

Provided that $I_r J_s - J_s I_r = 0$, the locally defined functions b_r, c_s, e_r, e_{sr} can be expressed, using (2.5), in terms of a and the complex structures I_r and J_s as follows:

$$\begin{aligned} \partial_i b_r &= -I_r^j{}_i \partial_j a, & \partial_i c_s &= -J_s^j{}_i \partial_j a \\ \partial_i e_r &= \Pi_{rr}^j{}_i \partial_j a, & \partial_i e_{sr} &= \Pi_{sr}^j{}_i \partial_j a, \end{aligned} \quad (6.2)$$

where $\Pi_{sr} := I_r J_s$ are product structures. The Bogomol'nyi equations are then

$$\begin{aligned} \partial_t \phi^i &= 0, \\ \partial_x \phi^i &= m [\alpha_+ \alpha_- g^{ij} + \alpha_+ \beta_-^s J_s^{ij} + \alpha_- \beta_+^r I_r^{ij} + \beta_+^r \beta_-^s \Pi_{sr}^{ij}] \partial_j a. \end{aligned} \quad (6.3)$$

This model has 16 topological charges each of which is associated with one of the functions a, b_r, c_r, e_r, e_{sr} , but all the Noether charges are zero. The values of the topological charges for soliton configurations are the differences of the values of these functions at the vacua between which the soliton interpolates. In distinction to the (2,2) models, one cannot choose coordinates on the target manifold of a generic (4,4) model such that all complex structures are simultaneously constant; for this reason the Bogomol'nyi equations (6.3) cannot, in general, easily be further simplified.

We shall now specialize to the case that $H = 0$. We may then choose $I_r = J_r$, set $T_r = Z_{rr}$ and introduce functions a, b_r, c_s, e_r, e_{sr} as in (6.1). The conditions (2.5) imply that

$$\begin{aligned} X &= T_r, & Y_s &= 0, & Z_r &= 0, & Z_{sr} &= 0 \quad s \neq r, \\ c_r &= -b_r, & a &= 0, & e_r &= 0, & e_{sr} &= 0, \\ X_i &= I_r^j{}_i \partial_j b_r; & r &= 1, 2, 3. \end{aligned} \quad (6.4)$$

The last of these equations implies that the vector field X is tri-holomorphic

($\mathcal{L}_X I_r = 0$). The Bogomol'nyi equations can now be written as

$$\begin{aligned}\partial_t \phi^i &= \frac{m}{2} \eta_0 X^i, \\ \partial_x \phi^i &= \frac{m}{2} \sum_r \eta_r I_r^i{}_j X^j\end{aligned}\tag{6.5}$$

where

$$\begin{aligned}\eta_0 &= 2(\alpha_+ a_-, 0, \dots, 0, \beta_{+1} \beta_{-1}, \beta_{+2} \beta_{-2}, \beta_{+3} \beta_{-3}) \\ \eta_r &= 2(0, \dots, 0, -\alpha_+ \beta_{-r}, \beta_{+r} \alpha_-, 0, \dots, 0)\end{aligned}\tag{6.6}$$

are 16-component vectors and the inner product is defined in eqn. (3.4). Because of equations (3.2) and the definition of inner product, we find that

$$\eta_0 \cdot \eta_0 + \sum_r \eta_r \cdot \eta_r = 1, \tag{6.7}$$

and that η_0 and η_r , $r = 1, 2, 3$, are mutually orthogonal. Because of the latter property we can change the basis in the 16-dimensional vector space such that the basis vectors include η_0 and η_r , which span a 4-dimensional Euclidean subspace.

As for the (1,1) models, when the torsion vanishes each Noether charge Q can be separated into a new Noether charge Q' and a topological charge. Thus, the model under discussion has one Noether charge Q'_X and three topological charges $Q_r \equiv Q_{b_r}$ associated with the functions b_r . These four charges can be viewed as a 4-vector in the four-dimensional space spanned by η_0 and η_r . Clearly, only this four-dimensional subspace of the original 16-dimensional space is relevant to the analysis to follow.

There is an alternative way to view the topological charges of the theory using symplectic geometry. To every complex structure I_r is associated the Kähler two-form

$$\Omega_{rij} := -g_{ik} I_r^k{}_j. \tag{6.8}$$

The vector field X is a Hamiltonian vector field of the symplectic form Ω_r , so

$$i_X \Omega_r = db_r, \tag{6.9}$$

i.e. b_r is the Hamiltonian function of X with respect to Ω_r . By analogy with the Killing potential of a holomorphic Killing vector field we shall call b_r the hyper-Kähler Killing potential. The topological charges are

$$Q_r = \int dx \phi^* (i_X \Omega_r) = \int dx \partial_x b_r . \quad (6.10)$$

To construct examples of zero torsion (4,4) massive supersymmetric sigma models with a $4k$ -dimensional hyper-Kähler target space, let the $4k$ coordinates be $\phi^i = (\varphi^\rho, \phi_r^\rho)$ where $\rho = 1, \dots, k$ and $r = 1, 2, 3$. Using these co-ordinates, we introduce the orthonormal frame

$$E^{0\rho} = (U^{-1/2})^{\rho\tau} (d\varphi^\tau + \omega_{r\lambda}^\tau d\phi^{r\lambda}) \quad E^{r\rho} = (U^{1/2})^{\rho\tau} d\phi^{r\tau} \quad (6.11)$$

and its dual

$$E_{0\rho} = (U^{1/2})^{\rho\tau} \frac{\partial}{\partial \varphi^\tau} \quad E_{r\rho} = (U^{-1/2})^{\rho\tau} \left(\frac{\partial}{\partial \phi^{r\tau}} - \omega_{r\lambda}^\tau \frac{\partial}{\partial \varphi^\lambda} \right) , \quad (6.12)$$

where $U^{\rho\tau}$ and $\omega^{\rho\tau}$ are the components of $k \times k$ symmetric matrix functions of ϕ_r^ρ . The metric takes the form [11]

$$\begin{aligned} dS^2 &\equiv \delta_{ab} \delta_{\rho\mu} E^{a\rho} \otimes E^{b\mu} \\ &= U^{\rho\tau} d\phi^\rho \cdot d\phi^\tau + (U^{-1})^{\rho\tau} (d\varphi^\rho + \omega^{\rho\lambda} \cdot d\phi^\lambda) (d\varphi^\tau + \omega^{\tau\mu} \cdot d\phi^\mu) , \end{aligned} \quad (6.13)$$

where $(a = (0, r), b = (0, s))$. The hyper-Kähler condition requires that U and ω satisfy

$$\frac{\partial}{\partial \phi_r^\rho} \omega_s^{\lambda\tau} - \frac{\partial}{\partial \phi_s^\lambda} \omega_r^{\rho\tau} = \varepsilon_{rst} \frac{\partial}{\partial \phi_t^\rho} U^{\lambda\tau} \quad \frac{\partial}{\partial \phi_r^\tau} U^{\rho\lambda} = \frac{\partial}{\partial \phi_r^\rho} U^{\tau\lambda} . \quad (6.14)$$

The three closed Kähler 2-forms are

$$\begin{aligned} \Omega_r &\equiv D^{(-)}(T_{0r})_{ab} \delta_{\rho\tau} E^{a\rho} \wedge E^{a\tau} \\ &= 2d\varphi^\rho \wedge d\phi_r^\rho + (2\omega_s^{\rho\tau} \delta_{tr} - U^{\rho\tau} \varepsilon_{rst}) d\phi_s^\rho \wedge d\phi_t^\tau , \end{aligned} \quad (6.15)$$

where

$$D^{(\pm)}(T_{ab}) = D(T_{ab}) \pm \frac{1}{2} \epsilon_{ab}{}^{cd} D(T_{cd}) , \quad (6.16)$$

and

$$D(T_{ab})^c{}_d = \delta_{ad} \delta_b{}^c - \delta_{bd} \delta_a{}^c , \quad (6.17)$$

i.e. $D^{(-)}(T_{0r})$ is the anti-self dual part of the fundamental representation of $\mathcal{L}(SO(4))$ restricted to one of the two $\mathcal{L}(SO(3))$ subalgebras of $\mathcal{L}(SO(4))$. Observe also that

$$D^{(\pm)}(T_{0r}) D^{(\pm)}(T_{0s}) = -\delta_{rs} \pm \epsilon_{rst} D^{(\pm)}(T_{0t}) . \quad (6.18)$$

The associated complex structures are

$$I_r = -D^{(-)}(T_{0r})^a{}_b \delta^\rho{}_\tau E^{b\tau} \otimes E_{a\rho} . \quad (6.19)$$

A basis for the k triholomorphic Killing vectors is $\left\{ \frac{\partial}{\partial \varphi^\rho} \right\}$. If we choose the particular linear combination

$$X = c^\rho \frac{\partial}{\partial \varphi^\rho} \quad (6.20)$$

for the Killing vector of the massive sigma-model (2.1) then it is easily seen that the hyper-Killing potential is just

$$b_r = c^\rho \phi_r^\rho . \quad (6.21)$$

The Bogomol'nyi equations (6.5) now reduce to

$$\begin{aligned} \partial_t \varphi^\rho &= \frac{m}{2} \eta_0 c^\rho & \partial_t \phi_r^\rho &= 0 \\ \partial_x \varphi^\rho &= -\frac{m}{2} c_\sigma (U^{-1})^{\sigma\tau} \eta^r \omega_{r\tau}^\rho & \partial_x \phi_r^\rho &= \frac{m}{2} (U^{-1})^{\rho\tau} c_\tau \eta_r . \end{aligned} \quad (6.22)$$

Using the equation for ϕ_r^ρ the equation for φ can be rewritten as

$$\partial_x \varphi^\rho = -\partial_x \phi^{r\tau} \omega_{r\tau}^\rho . \quad (6.23)$$

The first two Bogomol'nyi equations of (6.22) can be solved easily as follows:

$$\varphi^\rho(x, t) = \frac{m}{2} \eta_0 c^\rho t + \varphi^\rho(x) \quad \phi_r^\rho(x, t) = \phi_r^\rho(x) . \quad (6.24)$$

Substituting the solutions (6.23) into the other Bogomol'nyi equations, we get the following independent ordinary differential equations:

$$\frac{d}{dx} \phi_r^\rho(x) = \frac{m}{2} (U^{-1})^{\rho\tau} c_\tau \eta_r \quad \frac{d}{dx} \varphi^\rho(x) = -\frac{d}{dx} \phi^{r\tau} \omega_{r\tau}^\rho . \quad (6.25)$$

The only non-trivial differential equation to be integrated is that for ϕ_r^ρ . This is because if the solution for ϕ_r^ρ is known then the solution of the equation for $\varphi^\rho(x)$ can be found from

$$\varphi^\rho(x) = -\int dx \frac{d}{dx} \phi^{r\tau} \omega_{r\tau}^\rho + \varphi_0^\rho \quad (6.26)$$

where φ_0^ρ is a constant, i.e $\varphi^\rho(x)$ is the holonomy of the connection $\omega_{r\tau}^\rho$. Note that $\omega_{r\tau}^\rho$ depends only on ϕ_r^ρ .

When the target space is four-dimensional the metrics of (6.13) are the Gibbons-Hawking multi centre 4-metrics [8]

$$dS^2 = U d\phi \cdot d\phi + U^{-1} (d\varphi + \omega \cdot d\phi)^2 \quad (6.27)$$

where φ is an angular variable, ω and ϕ are now considered as vectors in Euclidean three-space, and $\text{curl } \omega = \text{grad } U$. An explicit soliton solution was given in [3] for the special case of the two-centre metric

$$U = 2\mu \left[\frac{1}{|\phi - \phi_0|} + \frac{1}{|\phi + \phi_0|} \right] \quad (6.28)$$

($\mu > 0$, $0 \leq \varphi \leq 8\pi\mu$), and triholomorphic Killing vector $X = \frac{\partial}{\partial \varphi}$ (so that $V = \frac{m^2}{4} U^{-1}$). We conclude by presenting some further aspects of this solution. A

solution of $\text{curl } \omega = \text{grad } U$ for U given by (6.28) is

$$\begin{aligned} \omega = & \mu(\phi \times \phi_0) \left\{ \frac{|\phi_0| - |\phi - \phi_0|}{|\phi|^2 |\phi_0| |\phi - \phi_0|} - \frac{|\phi_0| - |\phi + \phi_0|}{|\phi|^2 |\phi_0| |\phi + \phi_0|} \right. \\ & + \frac{\phi \cdot \phi_0}{|\phi|^2} \left[\frac{1}{(|\phi| |\phi - \phi_0| + \phi \cdot (\phi - \phi_0)) |\phi - \phi_0|} - \frac{1}{(|\phi| |\phi - \phi_0| - \phi \cdot \phi_0) |\phi_0|} \right. \\ & \left. \left. + \frac{1}{(|\phi| |\phi + \phi_0| + \phi \cdot (\phi + \phi_0)) |\phi + \phi_0|} - \frac{1}{(|\phi| |\phi + \phi_0| + \phi \cdot \phi_0) |\phi_0|} \right] \right\}. \end{aligned} \quad (6.29)$$

Note that this ω is orthogonal to ϕ_0 . Since $Q = 2\phi_0$ for this example, and η is parallel to Q for a solution saturating the Bogomol'nyi bound, we conclude that $\eta \cdot \omega = 0$ and (6.26) reduces to $\varphi(x) = \varphi_0$. The solution for ϕ is straightforward and is

$$\phi^r = \phi_0^r \tanh \left(\frac{m|\eta|}{8\mu} (x - x_0) \right). \quad (6.30)$$

This is the ‘Q-kink’ solution found in [3]. Note that ω is determined only up to the addition of the gradient of a scalar function. For a generic solution of $\text{curl } \omega = \text{grad } U$, the 3-vector ω would not be orthogonal to η , and φ would not be constant. However, it is possible to redefine φ to compensate for a gradient term in ω and this means that, at least for the two-centre metric, we may always arrange for φ to be constant for the soliton solution.

The moduli space of this soliton solution is $\mathcal{R} \times S^1$ with co-ordinates (x_0, φ_0) . The metric on this space may be computed from (3.9) and is

$$ds^2 = \frac{1}{2} |\eta|^2 \left(\frac{m|\phi_0|}{|\eta|} dx_0^2 + \frac{4|\phi_0|}{m|\eta|^3} d\varphi_0^2 \right). \quad (6.31)$$

Defining the new variable y by

$$\varphi_0 = \frac{1}{2} m y, \quad (6.32)$$

and introducing the notation

$$\gamma \equiv |\eta|^{-1} = \frac{1}{\sqrt{1 - \eta_0^2}}, \quad M = m|\phi_0|, \quad (6.33)$$

we find that

$$\gamma^2 \left(\frac{ds}{dt} \right)^2 = \frac{1}{2} \left[\gamma M \dot{x}_0^2 + \gamma^3 M \dot{y}^2 \right] . \quad (6.34)$$

If we interpret η_0 as the velocity of the particle in the y direction, then γ is the time dilation factor associated with this motion, so $Q_X = \gamma \eta_0 |Q|$ is the momentum in this direction and the total energy $E = \gamma M$ is the inertial mass for acceleration in orthogonal directions, i.e. in the x_0 direction. In this interpretation the inertial mass in the y direction would be expected to be $\gamma^3 M$ and this is precisely the coefficient of the \dot{y}^2 term in (6.34). Thus, the right hand side of (6.34) may be interpreted as the bosonic part of the soliton's full effective worldline action which will have an $N = 4$ worldline supersymmetry. Note that the left hand side of (6.34) can be rewritten in terms of the soliton's proper time τ , defined by $d\tau = \gamma^{-1} dt$, as $(ds/d\tau)^2$.

This ‘Kaluza-Klein’ interpretation can be understood as a consequence of the fact that this massive sigma model model can be obtained by (non-trivial) dimensional reduction of a massless model in three dimensions, as explained in [3]. In contrast, because of the torsion, the massive (1,1) model of section 4 cannot be so obtained and the effective Lagrangian (4.35) is not of the same form. Indeed, while it is tempting to identify κ in (4.33) as the velocity in the extra dimension this identification is not consistent with the interpretation of Q_X as the momentum in this direction since Q_X does not vanish for $\kappa = 0$. Also, whereas the massive (4,4) sigma model action does not depend on the ‘velocity’ η_0 , the massive (1,1) sigma model action does depend on κ , which is therefore fixed by the model.

7. Summary

In previous work we have established the conditions required for (p,q) supersymmetry of the most general ‘massive’ supersymmetric sigma-model, by which we mean the inclusion into a massless model of a potential term $V(\phi)$ for the sigma-model scalar fields ϕ , and the concomitant Yukawa couplings. The question then arises of whether massive models can be found satisfying these conditions. In the absence of torsion the conditions imposed by (p,q) supersymmetry are simpler than those for non-vanishing torsion, and various supersymmetric massive models with zero torsion had been constructed previously. In this paper we have constructed a new class of $(1,1)$ supersymmetric massive sigma models with a group manifold as the target space, and non-zero torsion; a subclass can be considered as a massive extension of the well-known supersymmetric WZW models. It is possible to show the existence of massive $(2,2)$ models with torsion but it seems that there are no massive $(2,2)$ supersymmetric models with a group manifold as the target space. We have also constructed a class of $(4,4)$ models with $4k$ -dimensional target spaces, but without torsion, that generalize previously constructed models based on the Gibbons-Hawking 4-metrics.

In many cases the potential V has multiple isolated zeros, which are necessarily (supersymmetric) ground states. A general feature of models with such potentials is the existence of soliton solutions interpolating between these distinct vacua. Using the (p,q) supersymmetry algebra of the supersymmetry charges obtained in our previous work, we have shown that the solitons saturate a Bogomol’nyi bound on the energy, in terms of the topological and Noether charges which occur in the supersymmetry algebra as central charges. In simple cases for which the potential V has just two isolated zeros, exact solutions can be found and we presented a new soliton solution of a particular massive $(1,1)$ sigma model with the group $SU(2)$ as its target space.

For the explicit soliton solutions considered we have also computed the mass and charges, determined the metric on the moduli space and the (bosonic part of

the) associated effective Lagrangian for one soliton. Models for which the potential V has multiple vacua will generically allow multi-soliton solutions, although these solutions will generally be time-dependent and need not be solutions of the Bogomol'nyi equations (e.g. the supersymmetric sine-Gordon model). We leave to the future the determination of which of the new models introduced here have this feature.

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